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On certain non-continuous functions and shape

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In the shape category of topological spaces a shape morphism is constructed by a system of maps (= continuous functions) ; it is, in general, not generated by a single map.
Hence we have the following questions :

Question 1. Is it possible that a kind of non-continuous function induces a shape morphism ?

Question 2. Can a shape equivalence be generated by a certain non-continuous function ?

Definition 1. Let X and Y be topological spaces. A function $f : X \to Y$ is a connectivity function if for any connected $C \subset X$, the graph $G(f|C)$ of $f|C$ is connected.

Definition 2. A function $f : X \to Y$ is almost continuous if for any open set $N \subset X \times Y$ containing $G(f)$ there is a continuous function $g : X \to Y$ such that $G(g) \subset N$.
These notions have been considered to generalize Brouwer's fixed point theorem (cf. Stallings[10]).

Each of the following is intermediate to answer the questions.

Proposition 1. Let $f : X \to Y$ be an almost continuous function between compact metric spaces. Then there are ANR-sequences $\underline{X} = \{X_i, p_{ij}\}$ and $\underline{Y} = \{Y_i, q_{ij}\}$ with limits X and Y, respectively, and a system $\underline{f} : \underline{X} \to \underline{Y}$ of almost continuous functions $f_i : X_i \to Y_i$ such that $f_i p_{ij} = q_{ij} f_j$ for $i \leqq j$.

Proof. There are ANR-sequences $\underline{X}$ and $\underline{Y}$ with limits X and Y, respectively, and each projection p_i, q_i surjective, since both X and Y are compact metric. Define a function $f_i : X_i \to Y_i$ for each i, by the formula $f_i p_i = q_i f$. The almost continuity of f implies that of f_i.

Proposition 2. A bijective connectivity function with connectivity inverse function does not induce a shape equivalence.

Proof. By the example of Stallings [10,p.262]. Let X be the circle represented as the real numbers mod 1. Define a function $f : X \to X \times X$ by the formula

$$f(x \bmod 1) = 1/x \bmod 1, \text{ where } 0 < x \leqq 1.$$

Let Y be the graph of f and $f^* : X \to Y$ such that

$$f^*(x) = (x, f(x)).$$

Then f^* is a bijection, and both f^* and f^{*-1} are connectivity functions, but $Sh(X) \neq Sh(Y)$, because their 1-dimensional Čech cohomology groups are different.

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